Linear Algebra II

03/04/2017, Monday, 14:00 – 17:00

You are **NOT** allowed to use any type of calculators.

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1 $(8 + 7 = 15 \text{ pts})$ **Gram-Schmidt process**

Consider the vector space $\mathbb{R}^4$ with the inner product

$$\langle x, y \rangle = x^T y.$$

Let $S \subset \mathbb{R}^4$ be the subspace given by

$$S = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(a) Apply the Gram-Schmidt process to obtain an orthonormal basis for S.

(b) Find the closest element in the subspace S to the vector

$$\begin{bmatrix} a \\ b \\ a \\ b \end{bmatrix}$$

where a and b are real numbers.

2 (15 pts) **Cayley-Hamilton theorem**

Consider the matrix

$$A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$$

where a is an arbitrary positive real constant By using the Cayley-Hamilton theorem, show that for all $n = 0, 1, 2, \ldots$ we have

$$A^{2n} = (-1)^n a^{2n} I$$

and

$$A^{2n+1} = (-1)^n a^{2n} A.$$

3 $(3 + 8 + 4 = 15 \text{ pts})$ **Singular value decomposition**

Consider the matrix

$$M = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}.$$

(a) Find the singular values of M.

(b) Find a singular value decomposition for M.

(c) Find the best rank 1 approximation of M.

4 $(5 + 5 + 5 = 15$ pts) **Unitary matrices and diagonalization**

A square complex matrix is called normal if $A^H A = AA^H$.

(a) Let A be a square complex matrix. Assume that there exists a unitary matrix U such that $U^H AU$ is a diagonal matrix. Prove that A is normal.

(b) Let T be an upper triangular matrix such that $A = UTU^H$. Prove that if A is normal, then also T is normal.

(c) Prove that if A is normal, then there exists a unitary matrix U such that $U^H AU$ is a diagonal matrix.

5 $(3 + 2 + 3 + 3 + 4 = 15$ pts) **Symmetric matrices and eigenvalues**

(a) Let A be a symmetric $n \times n$ matrix. Let λ be an eigenvalue of A. Show that the algebraic multiplicity of λ is equal to its geometric multiplicity.

Now consider the $n \times n$ matrix $A = (a_{ij})$ with all entries a_{ij} equal to 1.

(b) Determine the rank of A.

(c) Determine the dimension of $\ker(A)$.

(d) Show that 0 is an eigenvalue of A and determine its algebraic multiplicity.

(e) Determine all eigenvalues of A.

6 $(3 + 6 + 6 = 15$ pts) **Jordan canonical form**

Consider the matrix

$$M = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 2 \end{bmatrix}$$

where a and b are arbitrary real constants.

(a) Find all eigenvalues of M together with their algebraic multiplicities.

(b) For each eigenvalue, determine for each choice of a and b its geometric multiplicity.

(c) Determine for each choice of a and b the Jordan canonical form of M.

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10 pts free